# Propellant Requirements for Midcourse Velocity Corrections

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**This paper presents an analytic method of determining propellant requirements for interplanetary midcourse velocity corrections. The analysis assumes that a single interplanetary midcourse maneuver vector may be defined a priori by a set of linear, Gaussian statistics and derives the exact probability density function for the midcourse magnitude in terms of Horn's confluent hypergeometric series in two variables. It is shown that the propellant requirements may be uniquely determined by the trace of the a priori covariance matrix and the ratio of the eigenvalues of this covariance matrix. Resulting data for differing eigenvalue ratios are exhibited to provide ease in using the computational algorithm suggested by the derivation.**

## Introduction

THE accurate determination of propellant requirements for midcourse velocity corrections is an important problem in the preliminary design of an interplanetary spacecraft. Since weight allocated for propellant usually represents a significant portion of the total spacecraft configuration weight, it must be allocated correctly to permit maximum weight available for science instrument and other subsystem designs. A correct representation of the probability density function for midcourse corrections is needed to permit consistent tradeoffs between the ability to make midcourse corrections and the reliability of the entire spacecraft configuration.

Currently there are at least two general methods standardly employed for determining the propellant requirements for midcourse corrections. They may be conveniently divided into two categories—approximation techniques and Monte Carlo analyses. Each of these two methods is used in preliminary spacecraft design to determine the size of the midcourse velocity propellant, and each of them usually leads to conservative (too much propellant) allocations for midcourse maneuvers.

Of the currently used methods, the most rapid approximation technique is that given by Hoffman-Young.[1] Based on empirical data, they compute the mean and standard deviation for the midcourse magnitude distribution to an accuracy of $\pm 5\%$ using the eigenvalues of the midcourse velocity covariance matrix. They then assume that the velocity magnitude distribution may be characterized by a Gamma distribution and from this assumption compute the requisite probability integrals for sizing the propellant. Although their method is excellent for computation of the mean and standard deviation, the Gamma distribution assumption may lead to a 15% conservative error in the 99th percentile of the magnitude distribution.

Since the true analytic nature of the midcourse velocity magnitude density function has apparently been unknown until this time, many of the statistics of midcourse corrections used to size propellant have been computed using Monte Carlo analyses. Johnson et al.[2] present a detailed analysis of the Mariner '71 maneuvers that uses Monte Carlo techniques. There are two significant drawbacks to using Monte Carlo for preliminary design studies. First, if a number of separate trajectories representing a proposed mission must be investigated to determine the propellant requirements, then a Monte Carlo analysis of each trajectory is required. Second, and more important, the information of interest in a midcourse analysis is the distribution at the tails or upper percentiles, and an inordinately large number of Monte Carlo samples must be investigated before there is any accuracy at the tail of the sample distribution.

In this paper an exact analytic solution for the density function of the midcourse velocity magnitude is presented that will reduce some of the conservatism associated with midcourse propellant allocations. The solution is also amenable to rapid computation and could be used for preliminary design without undue computer time usage.

## Probabilistic Formulation of a Midcourse Maneuver

Let $P_k$ be the control covariance matrix representing the Gaussian deviations from a reference or nominal trajectory at the conclusion of the $k$th maneuver on an interplanetary mission. For $k = 0$, this would be the injection covariance matrix. If the next maneuver, $k + 1$, is to be implemented at some later time and the state transition matrix $\phi_{k+1,k}$ maps linear trajectory perturbations from one maneuver to the next, then the ensemble set of trajectory variations upon which the $k + 1$st midcourse maneuver must operate is given by

$$P_{k+1} = \phi_{k+1,k} P_k \phi_{k+1,k}{}^T \tag{1}$$

At the time of the $k + 1$st maneuver, any estimated deviation $\delta\hat{X}_{k+1}$ of the trajectory from the nominal, based upon the orbit determination, will be mapped into target errors and corrected according to the linear guidance equation

$$\Delta \mathbf{V}_{k+1} = \mathbf{\Gamma}_{k+1}\, \delta\hat{X}_{k+1} \tag{2}$$

where $\mathbf{\Gamma}_{k+1}$ is called the guidance matrix and contains the guidance law. Generally $\mathbf{\Gamma}$ is a $3 \times 6$ matrix of rank 2 or 3 depending upon whether or not the specific midcourse maneuver is correcting just spatial miss in the impact parameter plane (2-variable targeting) or spatial miss plus time of flight (3-variable targeting). Since the orbit determination algorithm is assumed to be unbiased, the a priori description of the $k + 1$st midcourse velocity covariance to be applied as a trajectory correction is given by

$$\mathbf{S}_{k+1} = \mathbf{\Gamma}_{k+1} P_{k+1} \mathbf{\Gamma}_{k+1}{}^T \tag{3}$$

where $\mathbf{S}$ represents a $3 \times 3$ covariance matrix of rank 2 or 3 depending upon the choice of guidance law discussed earlier. The matrix $\mathbf{S}$, however, only describes the midcourse vector

Presented as Paper 73-172 at the AIAA 11th Aerospace Sciences Meeting, Washington, D.C., January 10–12, 1973; submitted February 9, 1973; revision received August 24, 1973. The authors are indebted to M. Puryear for implementation of the algorithms suggested in this paper and for generation of the numerical data presented.

Index categories: Navigation, Control, and Guidance Theory; Spacecraft Navigation, Guidance, and Flight-Path Control; Spacecraft Propulsion Systems Integration.

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statistics; to determine the propellant requirements, the velocity magnitude statistics and probability density function must be computed.

From the previous discussion $\Delta\mathbf{V}$ is recognized as a vector random variable with three components of mean zero and a three-dimensional Gaussian distribution characterized by the covariance S. The magnitude of the vector $\Delta\mathbf{V}$ is given by

$$X = |\Delta\mathbf{V}| = (\Delta V_x^2 + \Delta V_y^2 + \Delta V_z^2)^{1/2} \tag{4}$$

where $\Delta V_x$, $\Delta V_y$ and $\Delta V_z$ are the three components of $\Delta\mathbf{V}$. The problem, therefore, is to derive the probability density function for the random variable $X$. It is assumed that the S matrix has been transformed into diagonal form to simplify the analysis. There is no loss of generality with this assumption since the expression in Eq. (4) is invariant under any orthonormal transformation. Furthermore, for any covariance S, there always exists an orthogonal matrix which diagonalizes S, leaving the eigenvalues of S along the principal diagonal.

The density function for the random variable $X$ will be computed by first determining the density function for an auxiliary variable, $Z$, where

$$Z = X^2 = \Delta V_x^2 + \Delta V_y^2 + \Delta V_z^2 \tag{5}$$

and then by using a well-known theorem from elementary probability theory[3]: if $g(Z)$ is the density function for $Z$ for all values $Z \geq 0$, then the density function for $X = (Z)^{1/2}$ is given by

$$f(X) = 2Xg(X^2) = 2Xg(Z) \quad \text{for all} \quad X \geq 0 \tag{6}$$

Since it is possible to transform $\Delta\mathbf{V}$ so that its components are independent and uncorrelated and since the characteristic function for the Gaussian random variable $\Delta V_x^2$ is known to be

$$\Phi_{\Delta V_x^2}(u) = (1 - 2\lambda_1 iu)^{-1/2} \tag{7}$$

where $\lambda_1$ is the eigenvalue of S associated with $\Delta V_x$ in the transformed system, the characteristic function for the random variable $Z$ is simply given by the product of the characteristic functions for each individual component, that is

$$\Phi_z(u) = (1 - 2\lambda_1 iu)^{-1/2}(1 - 2\lambda_2 iu)^{-1/2} \times (1 - 2\lambda_3 iu)^{-1/2} \tag{8}$$

is the characteristic function for the random variable $Z$ in terms of the eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$ of the original covariance S.

The desired intermediate density function for the auxiliary random variable $Z$ may then be found by using the Fourier inversion formula

$$f(Z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iuZ}\Phi_z(u)du \tag{9}$$

After laborious computations, which can involve either use of Riemann-Liouville fractional integrals[4-5] or transforming the exponential Fourier integral above into a Laplace transform, the probability density function for $Z$ becomes

$$f(Z) = \left(\frac{4\alpha\beta\gamma}{\pi}\right)^{1/2} Z^{1/2}e^{-\alpha Z} * \Phi_2(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};(\alpha-\beta)Z,(\alpha-\beta)Z) \tag{10}$$

where

$$\alpha = 1/2\lambda_1, \beta = 1/2\lambda_2, \quad \text{and} \quad \gamma = 1/2\lambda_3 \tag{11}$$

The function $\Phi_2$ in Eq. (10) is Horn's confluent hypergeometric series in two variables defined in Ref. 4. This series can be explicitly written as $\Phi_2(\frac{1}{2},\frac{1}{2};\frac{3}{2};(\alpha-\beta)Z, (\alpha-\gamma)Z) =$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\Gamma(r+\frac{1}{2})\Gamma(s+\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(r+s+\frac{3}{2})}\frac{(\alpha-\beta)^r}{r!}\frac{(\alpha-\gamma)^s}{s!}Z^{(r+s)} \tag{12}$$

where $\Gamma(r+\frac{1}{2})$, $\Gamma(s+\frac{1}{2})$, etc., are the gamma or generalized factorial function.

Using the theorem relating the density function for the square root to the above density gives the exact form for the midcourse velocity density function, namely

$$f(X) = (16\alpha\beta\gamma\pi)^{1/2}X^2e^{-\alpha x}\Phi_2(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};(\alpha-\beta)X^2,(\alpha-\beta)X^2) \tag{13}$$

This series is known to be uniformly convergent over any bounded interval; thus it can be readily integrated term by term to obtain the probability integrals.

If the guidance policy being used is a two variable policy, then the matrix S has rank two and the general density function simplifies considerably. For that case, where $\lambda_1$ and $\lambda_2$ are the nonzero eigenvalues of S, and the random variable $X$ is again defined by Eq. (4)

$$f(X) = (4\alpha\beta)^{1/2}Xe^{-1/2(\alpha-\beta)X^2}I_0[\tfrac{1}{2}(\alpha-\beta)X^2] \tag{14}$$

where $\alpha$ and $\beta$ are defined in Eq. (11) and $I_0$ is a modified Bessel function of zeroth order and first kind.

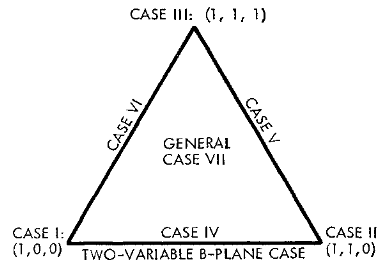


**Fig. 1 Schematic representation of possible eigenvalue ratios.**

## Case Description and Numerical Results

Using the probability density functions previously defined, the process of computing propellant allocations for midcourse maneuvers can be further simplified by recognizing some bounding characteristics that are functions of the eigenvalue ratios of the covariance matrix $S$. The triangle in Fig. 1 is a schematic representation of all possible ratios for the eigenvalues of $S$ and may be used both to define special cases of interest and the bounding conditions. For clarity in terminology, the following case representations are used in accompanying figures and in the Appendix.

The general case for a three-variable guidance correction is given by Case VII and the two-variable guidance policy is defined to be Case IV. It can be proved mathematically that the three vertices of the schematic triangle, which represent unidirectional, circular, and spherical distributions for the midcourse velocity vectors, represent the bounding conditions in terms of computing magnitudes of propellant requirements from the square root of the trace of the covariance matrix $S$. Each of these three special cases has a probability density function that is very easy to derive and integrate. The probability integrals are shown in Fig. 2.

Figure 2 shows the probability that the midcourse velocity magnitude is less than a certain multiple of the square root of the trace of the covariance $S$ for each of the three cases. Note that if the square root of the trace of $S$ is 100 m/sec, then the 99th percentile fuel requirement for the midcourse maneuver ranges from 194 m/sec to 258 m/sec, depending upon the ratio of the eigenvalues of $S$. Thus, any analytic

**Table 1 Case identification with eigenvalue ratios**

| Case Number | Eigenvalue ratio |
|---|---|
| I | (1,0,0) |
| II | (1,1,0) |
| III | (1,1,1) |
| IV | $(1,k^2,0)$, $0 < k^2 < 1$ |
| V | $(1,1,k^2)$, $0 < k^2 < 1$ |
| VI | $(1,k^2,k^2)$, $0 < k^2 < 1$ |
| VII | $(1,k^2,l^2)$, $0 < k^2 < 1, 0 < l^2 < 1$ |

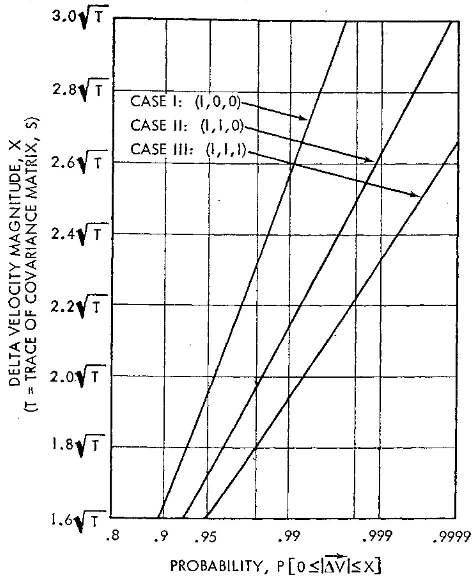


**Fig. 2 Cumulative distribution functions for cases I, II, and III.**

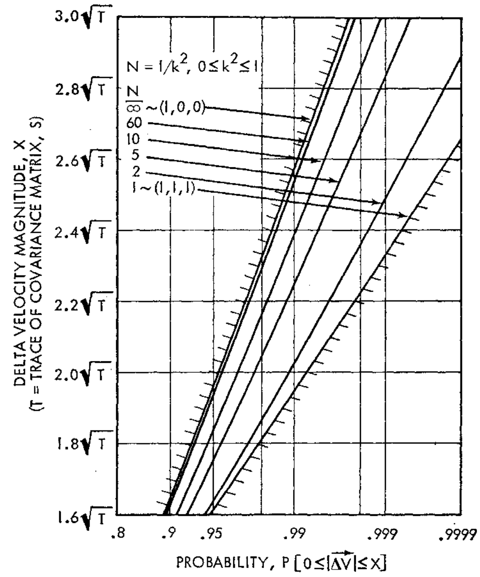


**Fig. 4 Cumulative distribution function for case VI: (1, $k^2$, $k^2$).**

method for midcourse propellant sizing that does not take into account the eigenvalue ratio and its effect on the underlying density function may be in error by as much as 30% at the 99th percentile.

Figure 3 presents the same kind of data, computed by integrating the density function in Eq. (14), for the general two-variable guidance policy (Case IV). If the probability integrals are normalized in terms of the square root of the trace of the covariance matrix $S$, then all the integrals are bounded by the two limiting cases (Case I and II) shown earlier. The individual cases in Fig. 3 are for differering values of the eigenvalue ratio and show the way in which the two limiting cases are approached.

Some results for Case VI are shown in Fig. 4, which applies to a general three-variable guidance policy when the two smaller eigenvalues of $S$ are equal. The figure is presented because the entire spectrum of midcourse sizing as a multiple of the square root of the trace is depicted. For example, if a midcourse maneuver has an eigenvalue ratio of $(1,\frac{1}{2},\frac{1}{2})$ and the square root of the trace is 100 m/sec, then the 99.9 percentile propellant requirement is 248 m/sec. If the trace remains the same but the eigenvalue ratio is changed to $(1,\frac{1}{10},\frac{1}{10})$, then the 99.9 percentile fuel requirement is 304 m/sec.

Figure 5 presents some additional data for the general case where all the eigenvalues are different. Again, the important points to be made are that the propellant allocation in terms of the square root of the trace are dependent only upon the eigenvalue ratios and that Cases I and III provide bounds for possible eigenvalue ratios.

## Summary and Conclusion

This paper has presented an exact analytic method for determining midcourse propellant allocations. The techniques are easy to implement on a digital computer and should remove both the propellant sizing conservatism associated with earlier approximate techniques and the computer time usage required to define the distribution tails by Monte Carlo analysis. The method is valid for calculation of the density function for the magnitude of a single interplanetary midcourse maneuver whose vector covariance matrix is Gaussian. The method should be extended to treat biased maneuvers (non-

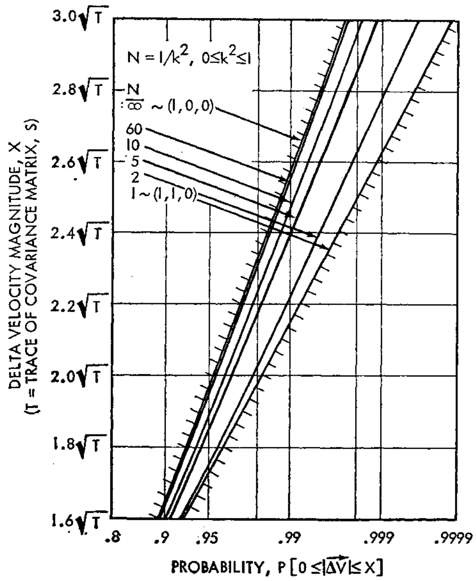


**Fig. 3 Cumulative distribution function for case IV: (1, $k^2$, 0).**

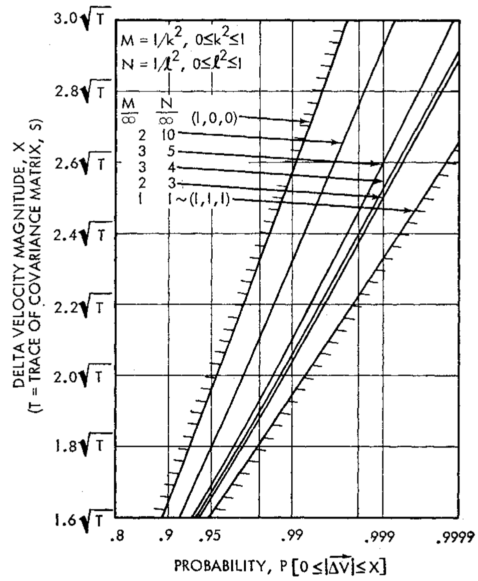


**Fig. 5 Cumulative distribution function for case VII: (1, $k^2$, $l^2$).**

zero vector mean) and multiple maneuvers, as well as maneuvers defined by non-Gaussian statistics.

## Appendix

In this appendix, statistical parameters of a midcourse velocity correction are computed with the aid of the probability density function presented in Eq. (13). Of particular interest, especially if a multiple midcourse maneuver analysis is to be performed, are the mean and standard deviation of the midcourse velocity distribution. Let $\mu_x$ and $\sigma_x$ be the mean and standard deviation of the random variable $X$. Then

$$\mu_x = \int_0^\infty Xf(X)dX \tag{15}$$

$$\mu_x = \left(\frac{16\alpha\beta\gamma}{\pi}\right)^{1/2}\int_0^\infty X^3 e^{-\alpha X^2}\Phi_2(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};(\alpha-\beta)X^2,(\alpha-\gamma)X^2)dX \tag{16}$$

Letting $Z = X^2$, Eq. (15) becomes

$$\mu_x = \left(\frac{4\alpha\beta\gamma}{\pi}\right)^{1/2}\int_0^\infty Ze^{-\alpha Z}\Phi_2[\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};(\alpha-\beta)Z,(\alpha-\gamma)Z]dZ$$

and the integrated function has the form

$$\mu_x = \left(\frac{4\beta\gamma}{\alpha^3\pi}\right)^{1/2} F_1\left[2;\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};\frac{\alpha-\beta}{\alpha},\frac{\alpha-\gamma}{\alpha}\right] \tag{18}$$

where $F_1[2;\frac{1}{2},\frac{1}{2};\frac{3}{2};\ (\alpha-\beta)/\alpha,\ (\alpha-\gamma)/\alpha]$ is Gauss' extended hypergeometric series in two variables[4–5] defined to be

$$F_1\left[2;\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{\alpha-\beta}{\alpha},\frac{\alpha-\gamma}{\alpha}\right] = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\Gamma(p+q+2)}{\Gamma(2)}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(\frac{1}{2})}\frac{\Gamma(q+\frac{1}{2})}{\Gamma(\frac{1}{2})}\frac{\Gamma(\frac{3}{2})}{\Gamma(p+q+\frac{3}{2})}\frac{1}{p!q!}\times\left(\frac{\alpha-\beta}{\alpha}\right)^p\left(\frac{\alpha-\gamma}{\alpha}\right)^q \tag{19}$$

The standard deviation follows readily from the mean as

$$\sigma_x = \{E[X^2]-\mu_x^2\}^{1/2} \quad \text{where} \quad E[X^2] = \int_0^\infty X^2 f(X)dX \tag{20}$$

Therefore

$$E[X^2] = \left(\frac{16\alpha\beta\gamma}{\pi}\right)^{1/2}\int_0^\infty X^4 e^{-\alpha X^2}\Phi_2(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};\ (\alpha-\beta)X^2,\ (\alpha-\gamma)X^2)dX \tag{21}$$

which becomes, after many laborious calculations

$$E[X^2] = \tfrac{1}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right) = \lambda_1+\lambda_2+\lambda_3 \tag{22}$$

From this and Eq. (18) it is a straightforward computation to determine the standard deviation.

The actual process of deriving the density function for the general case proceeded inductively, from the special cases to the general. The following formulas represent the results of all those derivations.

Case I: Eigenvalue ratios of (1,0,0).
$S$ has one eigenvalue $\lambda_1$ with $\alpha = 1/2\lambda_1$
Density for $X = |\Delta\mathbf{V}| : f(X) = (4\alpha/\pi)^{1/2}e^{-X^2\alpha}$
Mean Value $\mu_x$: $\mu_x = (1/\pi\alpha)^{1/2}$
Standard Deviation $\sigma_x$: $\sigma_x = ((\pi-2)/2\pi\alpha)^{1/2}$

Case II: Eigenvalue ratios of (1,1,0).
$S$ has two equal eigenvalues $\lambda_1 = \lambda_2$
Density for $X = |\Delta\mathbf{V}| : f(X) = 2\alpha Xe^{-X^2\alpha}$
Mean Value $\mu_x$: $\mu_x = (\pi/4a)^{1/2}$
Standard Deviation $\sigma_x$: $\sigma_x = [(4-\pi)/4\alpha]^{1/2}$

Case III: Eigenvalue ratios of (1,1,1).
$S$ has three equal eigenvalues $\lambda_1 = \lambda_2 = \lambda_3$
Density for $X = |\Delta\mathbf{V}| : f(X) = (16\alpha^3/\pi)^{1/2}X^2e^{-X^2\alpha}$
Mean Value $\mu_x$: $\mu_x = (4/\pi\alpha)^{1/2}$
Standard Deviation $\sigma_x$: $\sigma_x = [(3\pi-8)/2\pi\alpha]^{1/2}$

Case IV: Eigenvalue ratios of $(1,k^2,0)$, $0\le k^2\le 1$.
$S$ has two eigenvalues $\lambda_1$ and $\lambda_2$ with $\alpha = 1/2\lambda_1$ and $\beta = 1/2\lambda_2$
Density for $X = |\Delta\mathbf{V}|$:

$$f(X) = (4\alpha\beta)^{1/2}Xe^{-X^2\alpha}{}_1F_1[\tfrac{1}{2};1;(\alpha-\beta)X^2] = (4\alpha\beta)^{1/2}Xe^{-1/2(\alpha+\beta)X^2}I_0[\tfrac{1}{2}(\alpha-\beta)X^2]$$

Mean Value $\mu_x$:

$$\mu_x = \left(\frac{\pi}{4}\right)^{1/2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^{-1/2}{}_2F_1\left[\frac{1}{4},\frac{-1}{4};1;\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2\right]$$

Standard Deviation $\sigma_x$: $\sigma_x = [E(X^2)-\mu_x^2]^{1/2}$ where

$$E(X^2) = 1/2\alpha + 1/2\beta$$

Case V: Eigenvalue ratios of $(1,1,k^2)$, $0\le k^2\le 1$.
$S$ has two distinct eigenvalues $\lambda_1$ and $\lambda_2$ with $\alpha = 1/2\lambda_1$ and $\beta = 1/2\lambda_2$
Density for $X = |\Delta\mathbf{V}|$:

$$f(X) = \left(\frac{16\alpha^2\beta}{\pi}\right)^{1/2}X^2e^{-X^2\alpha}{}_1F_1[\tfrac{1}{2};\tfrac{3}{2};(\alpha-\beta)X^2]$$

Mean Value $\mu_x$:

$$\mu_x = \left(\frac{1}{\pi\beta}\right)^{1/2}\left[\left(\frac{\beta}{\alpha}\right)^{1/2}+\left(\frac{\alpha}{\alpha-\beta}\right)^{1/2}\arcsin\left(\frac{\alpha-\beta}{\alpha}\right)^{1/2}\right]$$

Standard Deviation $\sigma_x$: $\sigma_x = [E(X^2)-(\mu_x^2)^{1/2}]$

where

$$E(X^2) = 1/\alpha + 1/2\beta$$

Case VI: Eigenvalue ratios of $(1,k^2,k^2)$, $0\le k^2\le 1$.
$S$ has two distinct eigenvalues $\lambda_1$ and $\lambda_2$ with $\alpha = 1/2\lambda_1$ and $\beta = 1/2\lambda_2$
Density for $X = |\Delta\mathbf{V}|$:

$$f(X) = \left(\frac{16\alpha\beta^2}{\pi}\right)^{1/2}X^2e^{-X^2\alpha}{}_1F_1[1;\tfrac{3}{2};(\alpha-\beta)X^2]$$

Mean Value $\mu_x$:

$$\mu_x = \left(\frac{1}{\pi\alpha}\right)^{1/2}\left\{\left(\frac{\alpha}{\beta}\right)^{1/2}+\tfrac{1}{2}(\beta/\alpha-\beta)^{1/2}\times\ln\left[\frac{1+(1-\beta/\alpha)}{1-(1-\beta/\alpha)}\right]\right\}$$

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